MET O 11 TECHNICAL NOTE NO 76

Air-flow over a smooth mountain topography. Part 1

R. DIXON

1. Introduction.

The word "smooth" in the title implies that effects due to sharp peaks, cliffs, ravines, etc. are not considered. The geometry of the mountainous area is pictured as smooth in the sense that it is free of such singularities. It does not imply that frictional effects are absent.

This Note is the beginning of an attempt to acquire an understanding of the special effects which the existence of a smooth mountain topography will impose upon a flow field, and it is confined to establishing a few preliminary results. The approach is essentially intrinsic as it is the author's opinion that intrinsic coordinate systems are especially valuable for shedding light on the physical situation.

N.B. This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office branch.

2. Definitions and Notation

$\underline{V}$ denotes the three-dimensional wind velocity vector.
$\underline{t}$ is the unit vector in the direction of the trajectory, so that $\underline{V} = V\underline{t}$, where V is the speed
$\underline{n}$ is the unit normal vector to the trajectory.
$\underline{b}$ is the unit binormal vector.
$\underline{t}$ and $\underline{n}$ define the osculating plane
$\underline{n}$ and $\underline{b}$ define the normal plane.
$\underline{b}$ and $\underline{t}$ define the rectifying plane.
$\underline{\Omega}$ is the Earth's angular rotation vector
$f = 2\Omega \sin\phi$, where ϕ is the latitude
$\frac{d}{ds}$ indicates differentiation along a trajectory

κ_T is the 3-space trajectory curvature
$\bullet$ indicates the Gibbs scalar product
x indicates the Gibbs vector product
$\underline{k}$ is the unit vertical vector
V_H is the horizontal wind speed (a scalar)
K_T is the horizontal trajectory curvature

$\frac{\partial}{\partial n}$ indicates differentiation along an orthogonal to the horizontal trajectory

α is the specific volume.

The $\underline{t}$, $\underline{n}$, $\underline{b}$ triad is to be taken as a right-handed system so that

$$\underline{t} \times \underline{n} = \underline{b} \quad , \qquad \underline{n} \times \underline{b} = \underline{t} \quad , \quad \underline{b} \times \underline{t} = \underline{n} \tag{1}$$

and, of course

$$\underline{t} \bullet \underline{t} = \underline{n} \bullet \underline{n} = \underline{b} \bullet \underline{b} = 1 \quad , \qquad \underline{t} \bullet \underline{n} = \underline{t} \bullet \underline{b} = \underline{n} \bullet \underline{b} = 0 \tag{2}$$

$\underline{N}$ denotes a unit vector which is normal to $\underline{t}$ and lies in the tangent plane to the mountain surface.
$\underline{B}$ denotes a unit vector which is normal to the tangent plane to the mountain surface.

The triad $\underline{t}$, $\underline{N}$, $\underline{B}$ is also to be taken as a right-handed system, and therefore

$$\underline{t} \times \underline{N} = \underline{B} \quad , \qquad \underline{N} \times \underline{B} = \underline{t} \quad , \qquad \underline{B} \times \underline{t} = \underline{N} \tag{3}$$

and

$$\underline{t} \bullet \underline{t} = \underline{N} \bullet \underline{N} = \underline{B} \bullet \underline{B} = 1 \quad , \qquad \underline{t} \bullet \underline{N} = \underline{t} \bullet \underline{B} = \underline{N} \bullet \underline{B} = 0 \tag{4}$$

The situation is as depicted in Figure 1.

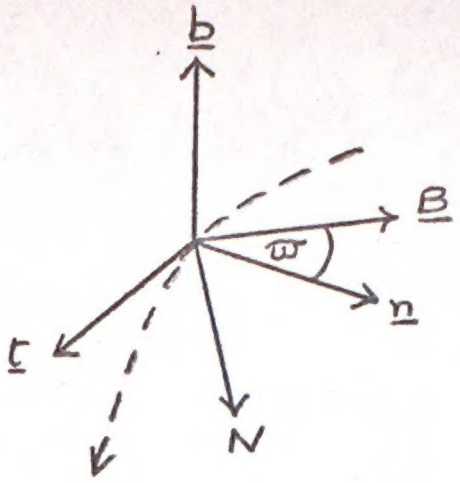

Figure 1

Note that $\underline{b}$, $\underline{n}$, $\underline{B}$ and $\underline{N}$ are all in the same plane, the Normal Plane. The angle ϖ, measured positive from $\underline{n}$ to $\underline{B}$, is called the Normal Angle, and the following relationships hold.

$$\underline{N} = \sin\varpi\,\underline{n} - \cos\varpi\,\underline{b} \tag{5}$$

$$\underline{B} = \cos\varpi\,\underline{n} + \sin\varpi\,\underline{b} \tag{6}$$

$$\underline{n} = \cos\varpi\,\underline{B} + \sin\varpi\,\underline{N} \tag{7}$$

$$\underline{b} = \sin\varpi\,\underline{B} - \cos\varpi\,\underline{N} \tag{8}$$

For motion on a plane $\cos\varpi = 0$ and $\sin\varpi = 1$ (9)

For motion along a geodesic $\cos\varpi = 1$ and $\sin\varpi = 0$ (10)

The fact that the flow is locally in the tangent plane to the mountain surface is expressed by the condition

$$\underline{V} \cdot \underline{B} = 0 \tag{11}$$

Note that the osculating Plane of the trajectory and the tangent plane to the mountain surface will not usually coincide.

Also $\underline{n}$ and $\underline{B}$ will be in the same or exactly opposite direction only if the air flows over the mountain along a geodesic. This is a characteristic property of a geodesic.

3. Motion along a Mountain Trajectory

The equation of motion may be written as

$$\frac{d\underline{V}}{dt} + 2\underline{\Omega} \times \underline{V} = \underline{F} \tag{12}$$

where $\underline{F}$ includes forces due to pressure gradient, gravity, friction, etc. The acceleration term in (12) can be written as

$$\frac{d\underline{V}}{dt} = V\frac{d\underline{V}}{ds} = V\frac{d(V\underline{t})}{ds} = V\frac{dV}{ds}\,\underline{t} + \frac{1}{\rho}V^2\underline{n} \tag{13}$$

which means that (12) can be rewritten as

$$\frac{V dV\underline{t}}{ds} + \varkappa_T V^2 \underline{n} + 2\underline{\Omega} \times V\underline{t} = \underline{F} \qquad (14)$$

and now taking $\underline{t}$, $\underline{N}$, and $\underline{B}$ in turn through (14) we get the three equations

$$\frac{V dV}{ds} = \underline{t} \cdot \underline{F} \qquad (15)$$

$$\varkappa_T \sin\varpi \; V^2 + 2(\underline{B} \cdot \underline{\Omega})V = \underline{N} \cdot \underline{F} \qquad (16)$$

$$\varkappa_T \cos\varpi \; V^2 - 2(\underline{N} \cdot \underline{\Omega})V = \underline{B} \cdot \underline{F} \qquad (17)$$

To obtain (15), (16), and (17) use has been made of (3) and (4) and also

$$\underline{N} \cdot \underline{n} = \sin\varpi \qquad (18)$$

$$\underline{B} \cdot \underline{n} = \cos\varpi \qquad (19)$$

The equations (15), (16) and (17) are the usual equations of motion in an unfamiliar and somewhat esoteric form. To connect them with more familiar matters consider plane flow. Using (9) these equations become

$$\frac{V dV}{ds} = \underline{t} \cdot \underline{F} \qquad (20)$$

$$\varkappa_T V^2 + 2(\underline{B} \cdot \underline{\Omega})\underline{V} = \underline{N} \cdot \underline{F} \qquad (21)$$

$$- 2(\underline{N} \cdot \underline{\Omega})V = \underline{B} \cdot \underline{F} \qquad (22)$$

Now specialize this to the case of horizontal plane flow. We then have $\underline{N} = \underline{n}$ and $\underline{B} = \underline{k}$, and the equations become

$$V_H \frac{dV_H}{ds} = \underline{t} \cdot \underline{F} \qquad (23)$$

$$K_T V_H^2 + 2(\underline{k} \cdot \underline{\Omega})V_H = \underline{n} \cdot \underline{F} \qquad (24)$$

$$- 2(\underline{n} \cdot \underline{\Omega})V_H = \underline{k} \cdot \underline{F} \qquad (25)$$

It is now seen that, with the usual conditions on $\underline{F}$, equation (24) is the gradient wind equation, whilst subject to scale considerations (not actually generally applicable in the case of mountain flow) equation (25) provides the hydrostatic equation. Further if the components of acceleration are zero (23) and (24) give the geostrophic relationship.

Intrinsic co-ordinate systems are not very convenient for computational purposes. They do have the merit that they can exhibit fundamental relationships between the flow parameters which may be quite awkward to obtain using the co-ordinate systems usually chosen for numerical work. A case in point is provided by equations (16) and (17). They are both quadratics in V and to make physical sense they must have a common solution. The condition for a common solution to exist is

$$\varkappa_T = \frac{[(\underline{B} \cdot 2\underline{\Omega})(\underline{B} \cdot \underline{F}) + (\underline{N} \cdot 2\underline{\Omega})(\underline{N} \cdot \underline{F})][\underline{n} \cdot 2\underline{\Omega}]}{(\underline{b} \cdot \underline{F})^2} \qquad (26)$$

Equation (26) is fundamental to the problem of flow over a smooth mountain topography because it determines the possible path over the mountain range by prescribing an exact, determined relationship between the force field, the mountain surface geometry (via $\underline{B}$ & $\underline{N}$) and the 3-space curvature of the path. In the case of horizontal, frictionless, hydrostatic flow (26) reduces to

$$K_T = - \frac{fV_H + \alpha \frac{\partial p}{\partial n}}{V_H^2} \qquad (27)$$

the gradient wind relationship.

If the flow follows a geodesic over the mountain surface then (26) becomes

$$\mathcal{H}_{\underline{T}} = \frac{[(\underline{B} \bullet \underline{F})(\underline{B} \bullet 2\underline{\Omega}) - (\underline{N} \bullet 2\underline{\Omega})(\underline{N} \bullet \underline{F})](\underline{B} \bullet 2\underline{\Omega})}{(\underline{N} \bullet \underline{F})^2} \qquad (28)$$

It is not suggested a priori that geodesics are favoured paths for air flow over a mountain but a geodesic net on the mountain surface constitutes a useful reference system and it is possible to discuss the flow in terms of its departure from geodesic. In this connection it needs to be appreciated that (28) is a strong condition on the flow. This is best seen by applying (28) to consider the possibility of air flowing directly N - S over the crest of a simple smooth mountain range orientated E - W. It is found from (28) that this is possible only if there is a considerable adjustment of the horizontal surface pressure gradient and/or a considerable departure from the hydrostatic condition.

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Met O 11 TECHNICAL NOTE No 76

Air-flow over a smooth mountain topography. Part 2.

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1. Introduction

This note shows that the divergence of the velocity vector measured in the tangent plane at a point on a mountain surface has an especially direct relationship with the rate of surface strain dyadic introduced by Dishington in 1965. The relationship is simply derived and its equivalence to the traditional expression for surface divergence in terms of the fundamental magnitudes of the first order is established. The relevance of the rate of surface strain dyadic to an approach favoured by other research workers is pointed out.

R H DISHINGTON:- "Rate of Surface-Strain Tensor"
Amer. J. Phys., 33, 10, Oct 1965 pp 827-831

N.B. This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office Branch.

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2. Surface Divergence and Deformation

A topographical surface can be defined by a position vector $\underline{r}$ which is a function of two independent parameters ϕ, ψ. The surface grad operator is then

$$\nabla_s = H^{-2}\left[\frac{\partial \underline{r}}{\partial \phi}\cdot\left(G\frac{\partial}{\partial \phi} - F\frac{\partial}{\partial \psi}\right) + \frac{\partial \underline{r}}{\partial \psi}\cdot\left(E\frac{\partial}{\partial \psi} - F\frac{\partial}{\partial \phi}\right)\right] \tag{1}$$

where E, F, & G are the fundamental magnitudes of the first order which appear in the surface metric

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2 \tag{2}$$

and

$$H^2 = EG - F^2 \tag{3}$$

Using the operator (1) on the 3-d velocity vector $\underline{V}$ thus gives the surface divergence as

$$\mathrm{div}_s \underline{V} = \nabla_s \cdot \underline{V} = H^{-2}\left[\frac{\partial \underline{r}}{\partial \phi}\cdot\left(G\frac{\partial \underline{V}}{\partial \phi} - F\frac{\partial \underline{V}}{\partial \psi}\right) + \frac{\partial \underline{r}}{\partial \psi}\cdot\left(E\frac{\partial \underline{V}}{\partial \psi} - F\frac{\partial \underline{V}}{\partial \phi}\right)\right] \tag{4}$$

Thus far we have followed the standard approach of text-books on differential geometry, but if $\underline{B}$ is the normal to the surface then the system of vectors reciprocal to $\frac{\partial \underline{r}}{\partial \phi}, \frac{\partial \underline{r}}{\partial \psi}, \underline{B}$ is the system $\underline{l}$, $\underline{m}$, $\underline{B}$ where $\underline{l}$ and $\underline{m}$ are given by

$$\underline{l} = H^{-2}\left(G\frac{\partial \underline{r}}{\partial \phi} - F\frac{\partial \underline{r}}{\partial \psi}\right) \tag{5}$$

and

$$\underline{m} = H^{-2}\left(E\frac{\partial \underline{r}}{\partial \psi} - F\frac{\partial \underline{r}}{\partial \phi}\right) \tag{6}$$

and putting (5) and (6) in (4) gives

$$\mathrm{div}_s \underline{V} = \underline{l}\cdot\frac{\partial \underline{V}}{\partial \phi} + \underline{m}\cdot\frac{\partial \underline{V}}{\partial \psi} \tag{7}$$

Now note that

$$\underline{B} \cdot \frac{\partial \underline{V}}{\partial B} = \underline{B} \cdot \left[\underline{B} \cdot \nabla \underline{V} \right] = \underline{B} \cdot \nabla \underline{V} \cdot \underline{B} \tag{8}$$

where ∇ is the 3-d Grad operator. Adding (8) to (7) there results

$$\mathrm{div}_s \underline{V} = \underline{\ell} \cdot \frac{\partial \underline{V}}{\partial \phi} + \underline{m} \cdot \frac{\partial \underline{V}}{\partial \psi} + \underline{B} \cdot \frac{\partial \underline{V}}{\partial B} - \underline{B} \cdot \nabla \underline{V} \cdot \underline{B} \tag{9}$$

But the first three terms on the RHS together constitute the 3-d divergence and so (9) becomes

$$\mathrm{div}_s \underline{V} = \mathrm{div} \underline{V} - \underline{B} \cdot \nabla \underline{V} \cdot \underline{B} \tag{10}$$

The gradient dyadic $\nabla \underline{V}$ may be split into its symmetric deformation and antisymmetric vorticity parts

$$\nabla \underline{V} = \mathcal{D} + \underline{\Phi} \tag{11}$$

and because of the antisymmetric character of $\underline{\Phi}$

$$\underline{B} \cdot \underline{\Phi} \cdot \underline{B} = 0 \tag{12}$$

Thus (10) may be written

$$\mathrm{div}_s \underline{V} = \mathrm{div} \underline{V} - \underline{B} \cdot \mathcal{D} \cdot \underline{B} \tag{13}$$

or, by introducing the Idemfactor I

$$\mathrm{div}_s \underline{V} = \underline{B} \cdot \left[(\mathrm{div} \underline{V}) I - \mathcal{D} \right] \cdot \underline{B} \tag{14}$$

where the quantity $\left[(\mathrm{Div}\, \underline{V})\ I - D \right]$ is precisely the rate of surface strain dyadic ${}^{s}D$ introduced by Dishington in 1965, so that

$$\mathrm{div}_s \underline{V} = \underline{B} \cdot {}^{s}\mathcal{D} \cdot \underline{B} \tag{15}$$

Readers unfamiliar with this approach and having the same suspicious turn of mind as the author may wonder whether this gives the right answer for $\mathrm{Div}_H \underline{V}$ when the surface is horizontal. It does, because then ${}^{s}D$ has the explicit nonion form.

$$
{}^{S}D = \begin{bmatrix}
\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\underline{i}\,\underline{i} & -\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\underline{i}\,\underline{j} & -\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\underline{i}\,\underline{k} \\
-\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\underline{j}\,\underline{i} & \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\underline{j}\,\underline{j} & -\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\underline{j}\,\underline{k} \\
-\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\underline{k}\,\underline{i} & -\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\underline{k}\,\underline{j} & \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\underline{k}\,\underline{k}
\end{bmatrix} \qquad (16)
$$

The horizontal plane may be defined by two unit vectors $\underline{i}$ and $\underline{j}$, whilst the vertical unit vector is $\underline{k}$, so that in this case $\underline{B} = \underline{k}$. Without loss of generality $\underline{i}$, $\underline{j}$, $\underline{k}$ may be taken as a right-handed orthogonal system, and it is then clearly true that

$$
Div_H \underline{V} = \underline{k} \cdot {}^{S}D \cdot \underline{k} \qquad (17)
$$

Equation (15) shows that the mountain surface divergence is very simply related to the topography via the deformation dyadic ${}^{S}D$. It is likely therefore that ${}^{S}D$ is a parameter of fundamental importance in the study of topographical flow. Other authors have shed light on the topic by accounting for the vorticity budget in a column of air as it flows over a mountain. The probable importance of ${}^{S}D$ is reinforced by reiterating a point made by Dishington in his paper, namely that

$$
Div\, {}^{S}D = \tfrac{1}{2}\, Curl\, Curl\, \underline{V} \qquad (18)
$$

and consequently since Div Curl $\underline{V}$ is identically zero, the specification of Div ${}^{S}D$ inside a closed column determines the vorticity everywhere inside the column subject to boundary conditions. Additionally, the present writer draws attention to the fact that the vorticity equation itself can be written as

$$
\frac{d\underline{Q}}{dt} + \underline{Q} \cdot {}^{S}D = -\nabla\alpha \times \nabla p \qquad (19)
$$

where $\underline{Q}$ is the absolute vorticity vector

Further information on Dishington's Rate of Surface-Strain Tensor can be found in Met O 11 Tech Note No 77.

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February 1977

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MET O 11 TECHNICAL NOTE NO 76

AIR-FLOW OVER A SMOOTH MOUNTAIN TOPOGRAPHY - Part 3

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1. Introduction

Relationships connecting the tangent plane intrinsic components of vorticity and deformation with the geodesic torsion and normal curvature are derived. They are expressed by equations (17) and (18). They are derived by way of a kinematical and geometrical argument which leads to a general vector relationship, equation (12), which is consistent with the customery boundary assumption of no flow normal to the surface. Indeed the tangential relationship (17) and (18) are shown to be derivable from this basic boundary condition.

NB: This paper has not been published. Permission to quote from it must be obtained from the Assistant Director of the above Meteorological Office Branch.

2. Deformation, Vorticity, Normal Curvature, and Geodesic Torsion

In his 1967 paper Dishington established that the rate of change, along a trajectory, of the unit normal $\underline{B}$ to an arbitrary fluid surface element is given by

$$V \frac{d\underline{B}}{ds} = {}^{s}\mathcal{D} \cdot \underline{B} - (\underline{B} \cdot {}^{s}\mathcal{D} \cdot \underline{B})\underline{B} + \tfrac{1}{2} \mathrm{Curl}\, \underline{V} \times \underline{B} \qquad (1)$$

where

$\underline{V} = V \underline{t}$ the three-dimensional velocity vector.

V is the scalar magnitude of the velocity vector.

$\underline{t}$ is the unit vector in the direction of the trajectory.

$\frac{d}{ds}$ represents differentiation along the trajectory.

${}^{s}\mathcal{D}$ is Dishington's rate of surface strain tensor and is in fact a dyadic given by

$${}^{s}\mathcal{D} = (\mathrm{div}\, \underline{V})I - \mathcal{D} \qquad (2)$$

where

$\mathcal{D}$ is the deformation dyadic, the symmetric part of $\nabla \underline{V}$, and

I is the Idemfactor.

If now we consider an element of fluid area which is sliding over a smooth topography then $\underline{B}$ in this case can be identified with the normal to the topographical surface and, for a given trajectory, $\frac{d\underline{B}}{ds}$ will be expressible in terms of the topographical geometry. We need therefore to derive an expression for $\frac{d\underline{B}}{ds}$ on a given topographical surface. This

may be done as follows

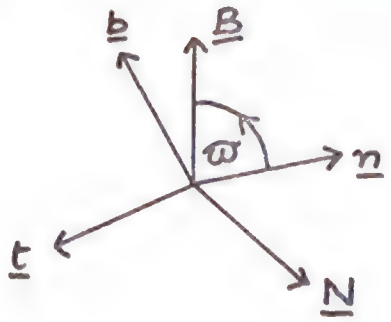

Figure 1.

The situation is as shown in Figure 1. $\underline{t}, \underline{n}, \underline{b}$ is the familiar space curve dextral triad, so that $\underline{t}$ and $\underline{n}$ define the osculating plane, $\underline{n}$ and $\underline{b}$ define the normal plane, and $\underline{b}$ and $\underline{t}$ define the rectifying plane. $\underline{B}$ is the normal to the topographical surface, and $\underline{N}$ is a unit vector lying in the tangent plane to the surface and it is orientated so that $\underline{t}, \underline{N}, \underline{B}$ is also a dextral triad. ϖ is the normal angle and it is measured positive from $\underline{n}$ to $\underline{b}$.

The angular velocity of the triad $\underline{t}, \underline{n}, \underline{b}$ as it moves with unit speed along the trajectory, considered as a curve in space, is given by the Darboux vector

$$\underline{\Delta} = T\underline{t} + \varkappa \underline{b} \qquad (3)$$

where T is the torsion of the trajectory,

and $\varkappa$ is the curvature of the trajectory.

The angular velocity of the triad $\underline{t}, \underline{N}, \underline{B}$ as it moves with unit speed along the trajectory considered as a curve on the topographical surface will consist of the angular velocity of the triad $\underline{t}, \underline{n}, \underline{b}$ plus an additional rotation $\frac{d\varpi}{ds}$ about $\underline{t}$. This follows since $\underline{t}$ is

common to both triads. Thus the angular velocity of $\underline{t}$, $\underline{N}$, $\underline{B}$ is

$$\underline{\Sigma} = \underline{\Delta} + \frac{d\varpi}{ds}\underline{t}$$

which by virtue of (3) is

$$\underline{\Sigma} = \left(T + \frac{d\varpi}{ds}\right)\underline{t} + \varkappa \underline{b} \tag{4}$$

There are also the relationships

$$\underline{n} = \cos\varpi\, \underline{B} + \sin\varpi\, \underline{N} \tag{5}$$

$$\underline{b} = \sin\varpi\, \underline{B} - \cos\varpi\, \underline{N} \tag{6}$$

so that by substituting for $\underline{b}$ in (4), using (6) there results

$$\underline{\Sigma} = \left(T + \frac{d\varpi}{ds}\right)\underline{t} + \varkappa\sin\varpi\, \underline{B} - \varkappa\cos\varpi\, \underline{N} \tag{7}$$

But $\left(T + \frac{d\varpi}{ds}\right) = T_g$, the geodesic torsion

$\varkappa \sin\varpi = \varkappa_g$, the geodesic curvature

and $\varkappa\cos\varpi = \varkappa_B$, the normal curvature

(See Weatherburn "Differential Geometry Vol.1" pp 105, and 109 for these relationships).

Thus (7) can be written as

$$\underline{\Sigma} = T_g\underline{t} + \varkappa_g \underline{B} - \varkappa_B \underline{N} \tag{8}$$

Now since $\underline{\Sigma}$ is the angular velocity vector for the triad $\underline{t}$, $\underline{N}$, $\underline{B}$ as it moves with unit speed along the trajectory it follows that for any vector $\underline{a}$ which is fixed in $\underline{t}$, $\underline{N}$, $\underline{B}$ we have

$$\frac{d\underline{a}}{ds} = \underline{\Sigma} \times \underline{a} \tag{9}$$

and if we specialize $\underline{a}$ to be $\underline{B}$ then

$$\frac{d\underline{B}}{ds} = \underline{\Sigma} \times \underline{B} \tag{10}$$

On evaluating the vector product on the R.H.S. of (10), using (8), there results

$$\frac{d\underline{B}}{ds} = -\varkappa_B \underline{t} - T_g \underline{N} \tag{11}$$

and thus finally from (11), using (1), we get

$$^{s}D\cdot\underline{B} - (\underline{B}\cdot{}^{s}D\cdot\underline{B})\underline{B} + \tfrac{1}{2}\,Curl\,\underline{V}\times\underline{B} = -\varkappa_B\underline{V} - V T_g \underline{N} \tag{12}$$

The significance of (12) is that it gives the relationship which must hold between the vorticity and deformation fields and the geometry of the topography. If the scalar products $\underline{B}\cdot$, $\underline{N}\cdot$, and $\underline{t}\cdot$ are taken successively through (12) we get the three component scalar equations

$$\underline{V}\cdot\underline{B} = 0 \tag{13}$$

$$\underline{N}\cdot{}^{s}D\cdot\underline{B} - \underline{t}\cdot\tfrac{1}{2}\,Curl\,\underline{V} = -V T_g \tag{14}$$

$$\underline{t}\cdot{}^{s}D\cdot\underline{B} + \underline{N}\cdot\tfrac{1}{2}\,Curl\,\underline{V} = -V\varkappa_B \tag{15}$$

If we put $\underline{\omega}$ for $\frac{1}{2}Curl\,\underline{V}$ then in terms of the components along the axes being used

$$\underline{\omega} = \omega_t \underline{t} + \omega_N \underline{N} + \omega_B \underline{B} \tag{16}$$

and it is evident that (14) and (15) may be regarded as expressions giving the $\underline{N}$ and $\underline{B}$ components of $\underline{\omega}$, i.e.

$$\omega_t = V T_g + \underline{N}\cdot{}^{s}D\cdot\underline{B} \tag{17}$$

$$\omega_N = -V\varkappa_B - \underline{t}\cdot{}^{s}D\cdot\underline{B} \tag{18}$$

The obvious attempt to find an expression for ω_B fails. By taking $\underline{B}\times$ through (12) from the right an expression for $\underline{\omega}$ is obtained, but on taking $\underline{B}\cdot$ through this expression in order to get ω_B both sides vanish identically. Thus the presence of the surface imposes a constraint on the $\underline{t}$ and $\underline{N}$ components of the vorticity but not on the $\underline{B}$ component, an intuitively acceptable result.

Since (12) is a relationship which has been obtained by kinematical and geometrical reasoning with respect to trajectories it might seem that (13), (14), (15) are independent equations and that (14) and (15) constitute additional boundary conditions. This is not

the case as (12), and hence (14) and (15), may be derived directly from the basic boundary condition (13). The quantities $\varkappa_B$ and T_g on the RHS of (12) have been introduced into this Note in connection with a derivation concerning trajectories but they are very much properties of the topography and in fact they will be the same for any curve having the same direction $\underline{t}$ in the tangent plane at the point P under consideration. Also since $\underline{B}$ is constant in time at a given point the distinction between $\frac{\partial}{\partial s}$ and $\frac{d}{ds}$ disappears when these operators are applied to $\underline{B}$. Thus the whole argument from eqn (2) to eqn (11) can be repeated with respect to streamlines and it will lead to precisely the same eqn (11) written in the form

$$\frac{\partial \underline{B}}{\partial s} = - \kappa_B \underline{t} - \tau_g \underline{N} \tag{19}$$

where κ and τ have been introduced in connection with streamlines but where in fact $\frac{\partial \underline{B}}{\partial s} \equiv \frac{d\underline{B}}{ds}$, $\kappa_B \equiv \varkappa_B$ and $\tau_g \equiv T_g$. Then since the condition (13) holds everywhere on the surface we have, from (13)

$$\nabla_s (\underline{V} \cdot \underline{B}) = \underline{0} \tag{20}$$

where ∇_s is the surface gradient operator and the subscript s is used to distinguish this operation from the full three dimensional ∇ operation. The relationships between the more common surface operators and their 3-d analogues as applied to the velocity vector are, for future references

$$\nabla_s \underline{V} = \nabla \underline{V} - \underline{B} (\underline{B} \cdot \nabla \underline{V}) \tag{21}$$

$$\mathrm{div}_s \underline{V} = \mathrm{div}\, \underline{V} - \underline{B} \cdot \underline{\underline{D}} \cdot \underline{B} \tag{22}$$

$$\mathrm{Curl}_s \underline{V} = \mathrm{Curl}\, \underline{V} - \underline{B} \times (\underline{B} \cdot \nabla \underline{V}) \tag{23}$$

From (20), by expansion

$$\underline{B} \cdot \nabla_s \underline{V} + \underline{V} \cdot \nabla_s \underline{B} + \underline{B} \times \mathrm{Curl}_s \underline{V} + \underline{V} \times \mathrm{Curl}_s \underline{B} = \underline{0} \tag{24}$$

But as ∇_s is the surface operator and $\underline{B}$ is the unit normal to the surface

$$\underline{B} \cdot \nabla_s \underline{V} = \underline{0} \tag{25}$$

$$\mathrm{Curl}_s \underline{B} = \underline{0} \tag{26}$$

and

$$\underline{V} \cdot \nabla_s \underline{B} = \underline{V} \cdot \nabla \underline{B} = V \frac{\partial \underline{B}}{\partial s} \tag{27}$$

and so (24) yields

$$\underline{B} \times \mathrm{Curl}_s \underline{V} = - V \frac{\partial \underline{B}}{\partial s} \tag{28}$$

But, using (23) and the standard formula of expansion for a triple vector product (28) becomes

$$\underline{B} \times \mathrm{Curl}\, \underline{V} + \underline{B} \cdot \nabla \underline{V} - (\underline{B} \cdot \nabla \underline{V} \cdot \underline{B}) \underline{B} = - V \frac{\partial \underline{B}}{\partial s} \tag{29}$$

which by virtue of the decomposition

$$\nabla \underline{V} = (\mathrm{div}\,\underline{V})I - {}^{S}\!D - I \times \tfrac{1}{2}\,\mathrm{Curl}\,\underline{V} \qquad (30)$$

is the same as

$$\underline{B}\cdot{}^{S}\!D - (\underline{B}\cdot{}^{S}\!D\cdot\underline{B})\underline{B} + \tfrac{1}{2}\,\mathrm{Curl}\,\underline{V}\times\underline{B} = V\frac{\partial \underline{B}}{\partial s} \qquad (31)$$

and in the light of (19) and the accompanying argument this is the same as (12).

3. Comment

Although the conditions on the tangent plane components of vorticty and deformation given by (17) and (18) stem from the basic boundary condition (13), as customarily used when simulating atmospheric flow by numerical models, these relationships have not been given explicitly before. Use of the basic condition (13) in a model may not of itself ensure that (17) and (18) hold in practice in all circumstances. Model performance in mountainous regions might be improved by enforcing (17) and (18) directly.

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Met O 11
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